Corrigé

Question I (2 + 4 = 6 points)

1) voir manuel EM66 à la page 55

2) voir manuel EM66 à la page 87

Question II (4 + 7 = 11 points)

1) $8e^{-3x} - e^{3x} \geq -7 | \cdot e^{3x} > 0$ $\boxed{\text{dom} = \mathbb{R}}$

$\Leftrightarrow 8 - e^{6x} \geq -7e^{3x}$

$\Leftrightarrow -e^{6x} + 7e^{3x} + 8 \geq 0 \quad (1)$

Posons $y = e^{3x} > 0$. (1) devient $-y^2 + 7y + 8 \geq 0$ $\Delta = 81$ $y_1 = -1$ et $y_2 = 8$

y	$-\infty$		-1		0		8		$+\infty$
$-y^2 + 7y + 8$		$-$	0	$+$		$+$	0	$-$	

Ainsi $0 < y \leq 8$, donc $e^{3x} \leq 8$ et par conséquent $x \leq \frac{\ln 2^3}{3} = \ln 2$.

$\boxed{S =]-\infty; \ln 2]}$

2) $x + \log_2(2^x - 0{,}5) = \log_{0,5} 9$

$\Leftrightarrow \log_2 2^x + \log_2(2^x - 0{,}5) = \frac{\log_2 9}{\log_2 0{,}5}$

$\Leftrightarrow \log_2 2^x(2^x - 0{,}5) = -\log_2 9$

$\Leftrightarrow \log_2\left(2^{2x} - \frac{1}{2} \cdot 2^x\right) = \log_2 \frac{1}{9}$

$\Leftrightarrow 2^{2x} - \frac{1}{2} \cdot 2^x - \frac{1}{9} = 0 \quad (2)$

CE: $2^x - 0{,}5 > 0$

$\Leftrightarrow 2^x > 2^{-1}$

$\Leftrightarrow x > -1$

$\boxed{\text{dom} =]-1; +\infty[}$

Posons $y = 2^x > 0$. (2) devient $y^2 - \frac{1}{2}y - \frac{1}{9} = 0$ $\Delta = \frac{25}{36}$

$y = \frac{2}{3}$ ou $y = -\frac{1}{6}$ (à rejeter, car $-\frac{1}{6} < 0$)

Ainsi $2^x = \frac{2}{3}$, donc $x = \log_2 \frac{2}{3} \in \text{dom}$ $\left(\log_2 \frac{2}{3} \approx -0{,}58\right)$.

$\boxed{S = \{1 - \log_2 3\}}$

Question III (5 + 5 + 2 + 8 = 20 points)

$$\boxed{f(x) = 1 - x - \ln\frac{x}{x-1}}$$

1)

CE et CD: $x - 1 \neq 0$ et $\dfrac{x}{x-1} > 0$

$\boxed{\text{dom} f = \text{dom}_{\text{d}} f = \left]-\infty; 0\right[\cup \left]1; +\infty\right[}$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(\underbrace{1-x}_{\to\mp\infty} - \underbrace{\ln\underbrace{\frac{x}{x-1}}_{\to 1}}_{\to 0}\right) = \mp\infty \qquad \boxed{\text{AO} \equiv y = -x+1}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(\underbrace{1-x}_{\to 0} - \underbrace{\ln\underbrace{\frac{\overbrace{x}^{\to 1}}{\underbrace{x-1}_{\to 0^+}}}_{\to +\infty}}_{\to +\infty}\right) = -\infty \qquad \boxed{\text{AVD} \equiv x = 1}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\underbrace{1-x}_{\to 1} - \underbrace{\ln\underbrace{\frac{\overbrace{x}^{\to 0^-}}{\underbrace{x-1}_{\to -1}}}_{\to 0^+}}_{\to -\infty}\right) = +\infty \qquad \boxed{\text{AVG} \equiv x = 0}$$

2)

$$f'(x) = -1 - \frac{x-1}{x}\cdot\frac{(x-1)-x}{(x-1)^2} = -1 + \frac{1}{x(x-1)} = \boxed{\frac{-x^2+x+1}{\underbrace{x(x-1)}_{>0}}}$$

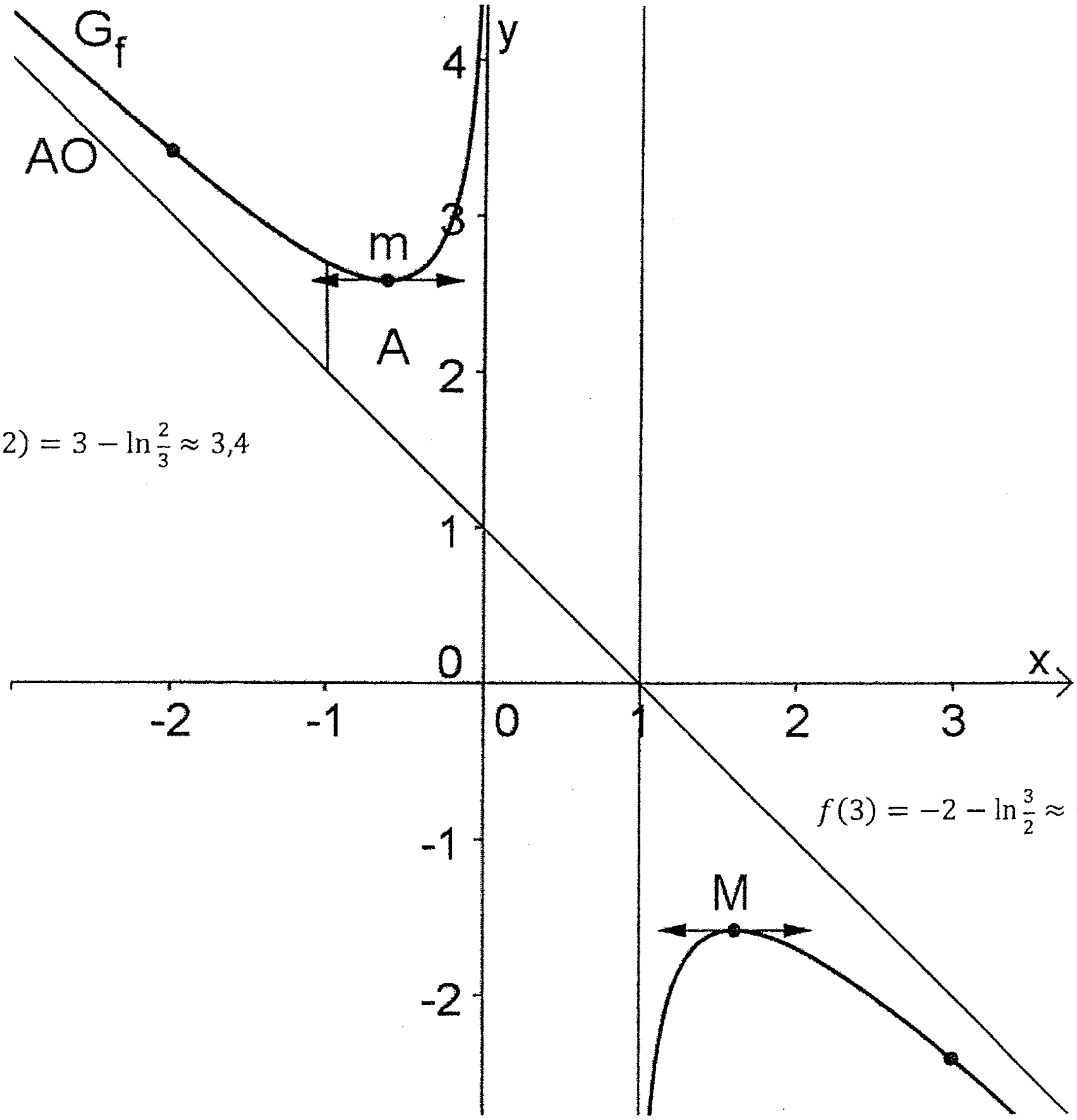
$\Delta = 1 + 4 = 5 \qquad x_{1/2} = \frac{-1\pm\sqrt{5}}{-2} < \begin{matrix}\approx -0{,}6\\ \approx \ \ 1{,}6\end{matrix}$

$$f''(x) = \left(-1 + \frac{1}{x(x-1)}\right)' = 0 - \frac{2x-1}{x^2(x-1)^2} = \boxed{\frac{1-2x}{\underbrace{x^2(x-1)^2}_{>0}}}$$

x	$-\infty$		$\frac{1-\sqrt{5}}{2}$		0	$\frac{1}{2}$	1		$\frac{1+\sqrt{5}}{2}$		$+\infty$
$f''(x)$		$+$		$+$		0		$-$		$-$	
$f'(x)$		$-$	0	$+$				$+$	0	$-$	
$f(x)$	$+\infty$	$\searrow$	$\approx 2{,}6$	$\nearrow$	$+\infty$		$-\infty$	$\nearrow$	$\approx -1{,}6$	$\searrow$	$-\infty$
G_f	AO	$\cup$	m	$\cup$	AVG		AVD	$\cap$	M	$\cap$	AO

$$\left(f\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{2} - \ln\frac{3-\sqrt{5}}{2} \approx 2{,}58 \text{ et } f\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1-\sqrt{5}}{2} - \ln\frac{3+\sqrt{5}}{2} \approx -1{,}58\right)$$

3)



$f(-2) = 3 - \ln\frac{2}{3} \approx 3{,}4$

$f(3) = -2 - \ln\frac{3}{2} \approx -2{,}4$

4) $\forall x \in]-\infty; 0[\quad \mathrm{F}(x) = \int [f(x) - (1-x)]dx = \int -\ln\frac{x}{x-1}dx = \int \ln\frac{x-1}{x}dx$

$$u(x) = \ln\frac{x-1}{x} \qquad u'(x) = \frac{x}{x-1}\cdot\frac{x-(x-1)}{x^2} = \frac{1}{x(x-1)}$$
$$v'(x) = 1 \qquad v(x) = x$$

$$\mathrm{F}(x) = x\ln\frac{x-1}{x} - \int\frac{1}{x-1}dx = x\ln\frac{x-1}{x} - \ln|x-1| + C = x\ln\frac{x-1}{x} - \ln(1-x) + C \quad (C \in \mathbb{R})$$

$\forall \lambda \in]-1; 0[\quad \mathrm{A}(\lambda) = \mathrm{F}(\lambda) - \mathrm{F}(-1) = \lambda\ln\frac{\lambda-1}{\lambda} - \ln(1-\lambda) - (-\ln 2 - \ln 2)$

$$= \boxed{\lambda\ln\frac{\lambda-1}{\lambda} - \ln(1-\lambda) + 2\ln 2}$$

$$\mathrm{A} = \lim_{\lambda\to 0^-}\mathrm{A}(\lambda) = \underbrace{\lim_{\lambda\to 0^-}\lambda\ln\frac{\lambda-1}{\lambda}}_{=\mathrm{L}} - \underbrace{\lim_{\lambda\to 0^-}\ln(1-\lambda)}_{=\ln 1 = 0} + 2\ln 2$$

$$\mathrm{L} = \lim_{\lambda\to 0^-}\overbrace{\lambda}^{\to 0^-}\ \overbrace{\ln\underbrace{\frac{\lambda-1}{\lambda}}_{\to+\infty}}^{\to+\infty}\ f.i. = \lim_{\lambda\to 0^-}\frac{\ln\frac{\lambda-1}{\lambda}}{\frac{1}{\lambda}} \overset{\boxed{\mathrm{H}}}{=} \lim_{\lambda\to 0^-}\frac{\frac{1}{\lambda(\lambda-1)}}{-\frac{1}{\lambda^2}} = \lim_{\lambda\to 0^-}\frac{-\lambda}{\lambda-1} = 0$$

$\mathrm{A} = \boxed{2\ln 2 \ \ u.a.}$ $(\approx 1{,}39\ u.a.;\ \mathrm{A} = 8\ln 2\ cm^2 \approx 5{,}55\ cm^2)$

Question IV (3 + 6 + 6 = 15 points)

1)

$$\lim_{x\to+\infty}\left(\frac{3x+1}{3x}\right)^{2x-3}=\lim_{x\to+\infty}\left(1+\frac{1}{3x}\right)^{2x-3}\underset{(y=3x)}{=}\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{\frac{2}{3}y-3}$$

$$=\lim_{y\to+\infty}\underbrace{\left[\left(1+\frac{1}{y}\right)^{y}\right]^{\frac{2}{3}}}_{\to e}\underbrace{\left(1+\frac{1}{y}\right)^{-3}}_{\to 1}=\boxed{\sqrt[3]{e^2}}\ (\approx 1{,}95)$$

2)

$$\int_{-2}^{2\sqrt{3}}\frac{3x+1}{\sqrt{16-x^2}}dx=-\frac{3}{2}\int_{-2}^{2\sqrt{3}}\frac{-2x}{\sqrt{16-x^2}}dx+\int_{-2}^{2\sqrt{3}}\frac{\frac{1}{4}}{\sqrt{1-\left(\frac{x}{4}\right)^2}}dx$$

$$=-3\int_{-2}^{2\sqrt{3}}\frac{(16-x^2)'}{2\sqrt{16-x^2}}dx+\int_{-2}^{2\sqrt{3}}\frac{\left(\frac{x}{4}\right)'}{\sqrt{1-\left(\frac{x}{4}\right)^2}}dx$$

$$=\left[-3\sqrt{16-x^2}+\arcsin\frac{x}{4}\right]_{-2}^{2\sqrt{3}}$$

$$=\left(-6+\frac{\pi}{3}\right)-\left(-6\sqrt{3}-\frac{\pi}{6}\right)$$

$$=\boxed{-6+\frac{\pi}{2}+6\sqrt{3}}\ (\approx 5{,}96)$$

3)

$$\int\frac{\sin 2x}{(1-2\sin^2 x)^4}dx=-\frac{1}{2}\int\frac{-4\sin x\cos x}{(1-2\sin^2 x)^4}dx$$

$$=-\frac{1}{2}\int(1-2\sin^2 x)'(1-2\sin^2 x)^{-4}dx$$

$$=-\frac{1}{2}\frac{(1-2\sin^2 x)^{-3}}{-3}+C$$

$$=\boxed{\frac{1}{6(1-2\sin^2 x)^3}+C}\ (C\in\mathbb{R})$$

CE: $1-2\sin^2 x\neq 0\Leftrightarrow \sin^2 x\neq\frac{1}{2}$

$$\Leftrightarrow \sin x\neq\frac{\sqrt{2}}{2}\text{ et }\sin x\neq-\frac{\sqrt{2}}{2}$$

$$\Leftrightarrow x\neq\frac{\pi}{4}+k\frac{\pi}{2}\quad(k\in\mathbb{Z})$$

p.ex. $\boxed{I=\left]-\frac{\pi}{4};\frac{\pi}{4}\right[}$

Question V (8 points)

Soit h la fonction définie par $h(x) = g(x) - f(x) = x^2 - \frac{9}{2}x + 2$. $\Delta = \frac{49}{4}$ $x_1 = \frac{1}{2}$ et $x_2 = 4$

x	$-\infty$		$\frac{1}{2}$		4		$+\infty$
$h(x)$		$+$	0	$-$	0	$+$	

Comme f est strictement croissante, $f(4) = -1 < 0$ et $h(x) < 0$ pour tout $x \in \left]\frac{1}{2}, 4\right[$,
$g(x) < f(x) < 0$ et $|g(x)| > |f(x)|$ pour tout $x \in \left]\frac{1}{2}, 4\right[$.

$$V = \pi \int_{\frac{1}{2}}^{4} ([g(x)]^2 - [f(x)]^2) dx$$

$$= \pi \int_{\frac{1}{2}}^{4} \left[(x^2 - 4x - 1)^2 - \left(\frac{1}{2}x - 3\right)^2\right] dx$$

$$= \pi \int_{\frac{1}{2}}^{4} \left(x^4 + 16x^2 + 1 - 8x^3 - 2x^2 + 8x - \frac{1}{4}x^2 + 3x - 9\right) dx$$

$$= \pi \int_{\frac{1}{2}}^{4} \left(x^4 - 8x^3 + \frac{55}{4}x^2 + 11x - 8\right) dx$$

$$= \pi \left[\frac{1}{5}x^5 - 2x^4 + \frac{55}{12}x^3 + \frac{11}{2}x^2 - 8x\right]_{\frac{1}{2}}^{4}$$

$$= \pi \left[\left(\frac{1024}{5} - 512 + \frac{880}{3} + 88 - 32\right) - \left(\frac{1}{160} - \frac{1}{8} + \frac{55}{96} + \frac{11}{8} - 4\right)\right]$$

$$= \pi \left(\frac{632}{15} + \frac{521}{240}\right)$$

$$= \boxed{\frac{10633}{240}\pi \ u.v.} (\approx 139{,}186 \ u.v.)$$