Examen de fin d'études secondaires.
Section $\boxed{B}$. Maths. II   10 juin 2002.

I

$$f : [-1, 1] \longrightarrow \mathbb{R}$$
$$x \longmapsto y = \text{Arc}\cos(4x^2+4x)$$

1) Conditions : $-1 \leq 4x^2+4x \leq 1$

$\Leftrightarrow$ (1) $4x^2+4x+1 \geq 0$ et $4x^2+4x-1 \leq 0$ $\Delta' = 8$

$\Leftrightarrow$ $(2x+1)^2 \geq 0$ et $x_1 = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$.

x		$-\frac{1}{2}-\frac{\sqrt{2}}{2}$		$-\frac{1}{2}+\frac{\sqrt{2}}{2}$	
$4x^2+4x-1$	$+$	0	$-$	0	$+$

$\Rightarrow x \in \left[-\frac{1}{2}-\frac{\sqrt{2}}{2} ; -\frac{1}{2}+\frac{\sqrt{2}}{2}\right]$.

$\text{dom} f = \text{dom}_c f = \left[-\frac{1}{2}-\frac{\sqrt{2}}{2} ; -\frac{1}{2}+\frac{\sqrt{2}}{2}\right]$; $\text{dom}_d f = \left]-\frac{1}{2}-\frac{\sqrt{2}}{2} ; -\frac{1}{2}\right[\cup \left]-\frac{1}{2} ; -\frac{1}{2}+\frac{\sqrt{2}}{2}\right[$ (car $(2x-1)^2 > 0$).

2) $f'(x) = \dfrac{-(8x+4)}{\sqrt{1-(4x^2+4x)^2}} = \dfrac{-4(2x+1)}{\sqrt{(1+4x^2+4x)(1-4x^2-4x)}} = \dfrac{-4(2x+1)}{|2x+1|\sqrt{-4x^2-4x+1}}$ >0

c.à.d. $f'(x)$ a le signe de $-2x-1$.

Tableau de variation :

x		$-\frac{1}{2}-\frac{\sqrt{2}}{2}$		$-\frac{1}{2}$		$-\frac{1}{2}+\frac{\sqrt{2}}{2}$	
$f'(x)$	//////	‖	$+$	‖	$-$	‖	//////
$f(x)$	//////	0	$\nearrow$	π	$\searrow$	0	//////

$f\left(-\frac{1}{2}-\frac{\sqrt{2}}{2}\right) = \text{Arc}\cos\left[4 \cdot \frac{1+2\sqrt{2}+2}{4} + 4 \cdot \frac{-1-\sqrt{2}}{2}\right] = \text{Arc}\cos(3+2\sqrt{2}-2-2\sqrt{2}) = \text{Arc}\cos 1 = 0.$

$f\left(-\frac{1}{2}+\frac{\sqrt{2}}{2}\right) = \text{Arc}\cos\left[3-2\sqrt{2}-2+2\sqrt{2}\right] = \text{Arc}\cos 1 = 0.$

$f\left(-\frac{1}{2}\right) = \text{Arc}\cos(1-2) = \text{Arc}\cos(-1) = \pi.$

3) $\lim\limits_{x \to \left(-\frac{1}{2}-\frac{\sqrt{2}}{2}\right)^+} f'(x) = \lim\limits_{x \to \left(-\frac{1}{2}-\frac{\sqrt{2}}{2}\right)^+} \dfrac{-4(2x+1)}{-(2x+1)\sqrt{-4x^2-4x+1}} = \lim\limits_{x \to \left(-\frac{1}{2}-\frac{\sqrt{2}}{2}\right)^+} \dfrac{4}{\underbrace{\sqrt{-4x^2-4x+1}}_{\to 0^+}} = +\infty$

$\lim\limits_{x \to \left(-\frac{1}{2}+\frac{\sqrt{2}}{2}\right)^-} f'(x) = \lim\limits_{x \to \left(-\frac{1}{2}+\frac{\sqrt{2}}{2}\right)^-} \dfrac{-4(2x+1)}{(2x+1)\sqrt{-4x^2-4x+1}} = \lim\limits_{x \to \left(-\frac{1}{2}+\frac{\sqrt{2}}{2}\right)^-} \dfrac{-4}{\underbrace{\sqrt{-4x^2-4x+1}}_{\to 0^+}} = -\infty$

C_f admet une ½-tangente verticale à droite au pt. $A\left(-\frac{1}{2}-\frac{\sqrt{2}}{2}, 0\right)$ et une demi-tangente verticale à gauche au pt. $B\left(-\frac{1}{2}+\frac{\sqrt{2}}{2}, 0\right)$.

$\lim\limits_{x \to \left(-\frac{1}{2}\right)^-} f'(x) = \lim\limits_{x \to \left(-\frac{1}{2}\right)^-} \dfrac{+4}{\sqrt{-4x^2-4x+1}} = \dfrac{4}{\sqrt{2}} = 2\sqrt{2} \simeq 2{,}83$

$\lim\limits_{x \to \left(-\frac{1}{2}\right)^+} f'(x) = \lim\limits_{x \to \left(-\frac{1}{2}\right)^+} \dfrac{-4}{\sqrt{-4x^2-4x+1}} = -2\sqrt{2}.$

C_f admet au point $C\left(-\frac{1}{2}, \pi\right)$, anguleux, deux demi-tangentes de coefficients directeurs $2\sqrt{2}$ (à gauche), $-2\sqrt{2}$ (à droite).

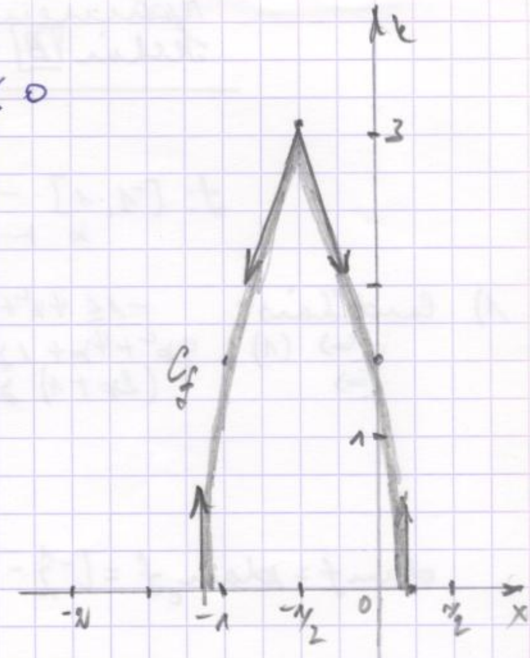
4) $\forall x \in \left]-\frac{1}{2}-\frac{\sqrt{2}}{2} ; -\frac{1}{2}\right[$: $f'(x) = \dfrac{4}{\sqrt{-4x^2-4x+1}}$

$$f''(x) = -4 \cdot \frac{\dfrac{-8x-4}{2\sqrt{-4x^2-4x+1}}}{-4x^2-4x+1} = \frac{16x+8}{\sqrt{(-4x^2-4x+1)^3}} < 0$$

$\forall x \in \left]-\frac{1}{2} ; -\frac{1}{2}+\frac{\sqrt{2}}{2}\right[$: $f'(x) = \dfrac{-4}{\sqrt{-4x^2-4x+1}}$

$$f''(x) = \frac{-16x-8}{\sqrt{(-4x^2-4x+1)^3}} < 0$$

x	$-\frac{1}{2}-\frac{\sqrt{2}}{2}$		$-\frac{1}{2}$		$-\frac{1}{2}+\frac{\sqrt{2}}{2}$
$f''(x)$		$-$	||	$-$	
C_f		$\cap$		$\cap$	



5) $f(0) = \text{Arc}\cos 0 = \frac{\pi}{2}$.

II

(E) $\quad \text{Arc}\sin x + \text{Arc}\sin 2x = \text{Arc}\cos 2x$.

- Conditions d'existence: 1) $\quad -1 \le x \le 1$
  2) $\quad -1 \le 2x \le 1 \Leftrightarrow -\frac{1}{2} \le x \le \frac{1}{2}$

finalement (E) existe $\forall x \in \left[-\frac{1}{2} ; \frac{1}{2}\right]$.

- Posons: $\text{Arc}\sin x = t_1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ c.à.d. $x = \sin t_1$
  $\text{Arc}\sin 2x = t_2 \in \left[-\frac{\pi}{2} ; \frac{\pi}{2}\right]$ c.à.d. $2x = \sin t_2$.
  $\text{Arc}\cos 2x = t_3 \in [0, \pi]$ c.à.d. $2x = \cos t_3$.

(E) : $\quad t_1 + t_2 = t_3$
$\Leftrightarrow \cos(t_1+t_2) = \cos t_3$
$\Leftrightarrow \cos t_1 . \cos t_2 - \sin t_1 . \sin t_2 = \cos t_3$

Or : $\quad \cos^2 t_1 = 1 - \sin^2 t_1 = 1 - x^2 \Rightarrow \cos t_1 = \sqrt{1-x^2}$ car $t_1 \in \left[-\frac{\pi}{2} ; \frac{\pi}{2}\right]$.

et : $\quad \cos^2 t_2 = 1 - \sin^2 t_2 = 1 - 4x^2 \Rightarrow \cos t_2 = \sqrt{1-4x^2}$ car $t_2 \in \left[-\frac{\pi}{2} ; \frac{\pi}{2}\right]$.

Remplaçons: (E) $\quad \sqrt{1-x^2} . \sqrt{1-4x^2} - x \cdot 2x = 2x$

$\Leftrightarrow \quad \sqrt{1-5x^2+4x^4} = 2x^2 + 2x \quad |$ carré

$\Leftrightarrow \quad 1 - 5x^2 + \cancel{4x^4} = \cancel{4x^4} + 8x^3 + 4x^2$

avec $2x(x+1) \ge 0$.

(E') $\quad 8x^3 + 9x^2 - 1 = 0$

avec :

x		-1		$-1/2$		0		$1/2$
$2x(x+1)$	$+$	0	$-$		$-$	0	$+$	

$x \in \left[0 ; \frac{1}{2}\right]$.

"racine" : $x = -1$.

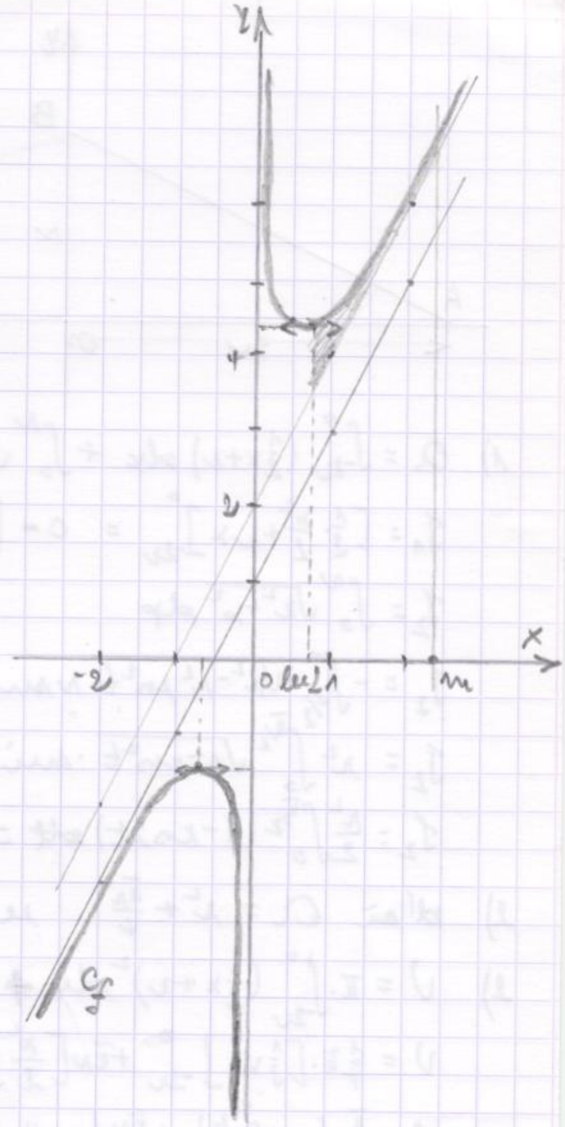
	8	9	0	-1
		-8	-1	1
-1	8	1	-1	0

finalement : (E') $\quad \underbrace{(x+1)}_{\neq 0}(8x^2 + x - 1) = 0$
$8x^2 + x - 1 = 0 \qquad \Delta = 33$.

$x_1 = \dfrac{-1+\sqrt{33}}{16} \simeq 0{,}297 \qquad x_2 = \dfrac{-1-\sqrt{33}}{16} < 0$ à écarter.

$S = \left\{ \dfrac{-1+\sqrt{33}}{16} \right\}$.

III



$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto y = 2x+1+\frac{e^x}{e^x-1}.$$

1) $\text{dom} f = \text{dom}_c f = \text{dom}_d f = \mathbb{R}^*$.

2) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\left[2x+1+\frac{e^x}{e^x(1-e^{-x})}\right] = +\infty$.

$\lim\limits_{x\to+\infty}\frac{f(x)}{x} = \lim\limits_{x\to+\infty}\left[2+\frac{1}{x}+\frac{1}{x\cdot(1-e^{-x})}\right] = 2$

$\lim\limits_{x\to+\infty}\left[f(x)-2x\right] = \lim\limits_{x\to+\infty}\left[1+\frac{1}{1-e^{-x}}\right] = 2$

A.O. $y = 2x+2$ lorsque $x \to +\infty$.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty}(2x+1)$ car $\lim\limits_{x\to-\infty}\frac{e^x}{e^x-1} = 0$.

A.O. $y = 2x+1$ lorsque $x\to-\infty$

$$\left.\begin{aligned}\lim_{x\to0^+} f(x) &= 1+\frac{1}{0^+} = +\infty\\ \lim_{x\to0^-} f(x) &= 1+\frac{1}{0^-} = -\infty\end{aligned}\right\}\ \text{A.V. } x=0.$$

3) $f'(x) = 2+\frac{e^x(e^x-1)-e^x\cdot e^x}{(e^x-1)^2} = \frac{2(e^x-1)^2-e^x}{(e^x-1)^2}$

$f'(x) = \frac{2\cdot e^{2x}-5\cdot e^x+2}{(e^x-1)^2}$

$f'(x)=0 \Leftrightarrow 2\cdot e^{2x}-5\cdot e^x+2=0 \qquad \Delta = 9$

$\Leftrightarrow e^x = 2$ ou $e^x = \frac{1}{2}$

$\Leftrightarrow x = \ln 2 \Rightarrow f(\ln 2) = 2\ln 2+3 \simeq 4{,}4$ m

$x = -\ln 2 \Rightarrow f(-\ln 2) = -2\ln 2 \simeq -1{,}4$.

4) représentation graphique.

5) $\mathcal{A}(m) = \int_{\ln 2}^{m}\left[f(x)-(2x+2)\right]dx = \int_{\ln 2}^{m}\left(\frac{e^x}{e^x-1}-1\right)dx = \left[\ln|e^x-1|\right]_{\ln 2}^{m}-\left[x\right]_{\ln 2}^{m}$

$\mathcal{A}(m) = \ln|e^m-1|-\ln(2-1)-m+\ln 2 = \ln(e^m-1)-m+\ln 2$ u.a.

$\mathcal{A}(m) = \ln e^m\left(1-\frac{1}{e^m}\right)-m+\ln 2 = \ln\left(1-\frac{1}{e^m}\right)+\ln 2.$

6) $\lim\limits_{m\to+\infty}\mathcal{A}(m) = \lim\limits_{m\to+\infty}\left[\ln(1-e^{-m})+\ln 2\right] = \ln 2$ u.a.

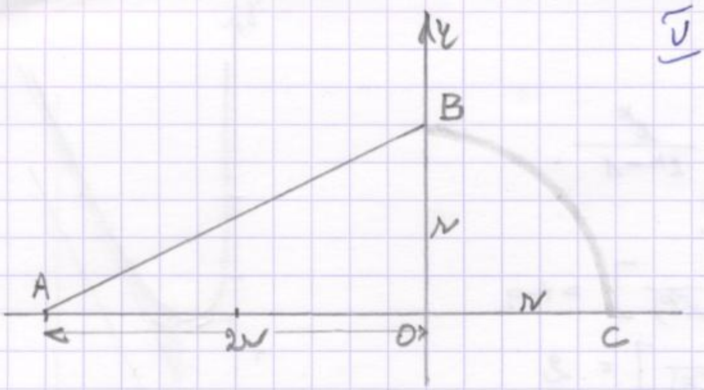
IV

$$J = \int_0^{\pi/4}\frac{dx}{1+2\cos^2 x} = \int_0^{\pi/4}\frac{dx}{1+\frac{2}{1+\tan^2 x}} = \int_0^{\pi/4}\frac{1+\tan^2 x}{3+\tan^2 x}\,dx = \int_0^{\pi/4}\frac{1}{3+tg^2 x}\cdot\frac{dx}{\cos^2 x}$$

Posons : $\sqrt{3}\cdot t = \tan x \Rightarrow \sqrt{3}\cdot dt = \frac{1}{\cos^2 x}\,dx \qquad x = 0 \Rightarrow t = 0;\ x = \frac{\pi}{4} \Rightarrow t = \frac{1}{\sqrt{3}}$

$$J = \sqrt{3}\cdot\int_0^{\frac{1}{\sqrt{3}}}\frac{1}{3+3t^2}\,dt = \frac{\sqrt{3}}{3}\int_0^{1/\sqrt{3}}\frac{dt}{1+t^2} = \frac{\sqrt{3}}{3}\left[\text{Arctan}\, t\right]_0^{1/\sqrt{3}} = \frac{\sqrt{3}}{3}\cdot\frac{\pi}{6} = \frac{\pi\sqrt{3}}{18}.$$

$\overline{\text{V}}$



Équation de (AB): $y = \frac{1}{2}x + r$

Équation de l'arc $\widehat{BC}$:

$x^2 + y^2 = r^2$

$y = \sqrt{r^2 - x^2}$ avec $y \geq 0$ et $0 \leq x \leq r$

1) $\mathcal{A} = \int_{-2r}^{0} \left(\frac{1}{2}x + r\right) dx + \int_{0}^{r} \sqrt{r^2 - x^2}\, dx = J_1 + J_2.$

$J_1 = \left[\frac{1}{2}\cdot\frac{x^2}{2} + rx\right]_{-2r}^{0} = 0 - \left[\frac{1}{4}\cdot 4r^2 - 2r^2\right] = r^2.$

$J_2 = \int_{0}^{r} \sqrt{r^2 - x^2}\, dx$ posons: $x = r\cos t$, $0 \leq t \leq \frac{\pi}{2}$

$dx = -r\sin t\, dt$

$x = 0 \Rightarrow t = \frac{\pi}{2};\quad x = r \Rightarrow t = 0$

$J_2 = -\int_{\pi/2}^{0} \sqrt{r^2 - r^2\cos^2 t}\cdot r\sin t\, dt$

$J_2 = r^2 \int_{0}^{\pi/2} \sqrt{1 - \cos^2 t}\cdot \sin t\, dt = r^2 \int_{0}^{\pi/2} \sin^2 t\, dt$ $|\sin t| = \sin t$ $(0 \leq t \leq \frac{\pi}{2})$

$J_2 = \frac{r^2}{2}\int_{0}^{\pi/2} (1 - \cos 2t)\, dt = \frac{1}{2}r^2\left[t\right]_{0}^{\pi/2} - \frac{1}{2}r^2\cdot\frac{1}{2}\left[\sin 2t\right]_{0}^{\pi/2} = \frac{1}{2}r^2\cdot\frac{\pi}{2} - \frac{1}{4}r^2\cdot 0 = \frac{\pi r^2}{4}$

d'où $\mathcal{A} = r^2 + \frac{\pi r^2}{4}$ u. a.

2) $V = \pi\cdot\int_{-2r}^{0} \left(\frac{1}{2}x + r\right)^2 dx + \pi\int_{0}^{r} (r^2 - x^2)\, dx = \pi\int_{-2r}^{0} \left(\frac{1}{4}x^2 + rx + r^2\right) dx + \pi\int_{0}^{r} (r^2 - x^2)\, dx$

$V = \frac{1}{4}\pi\cdot\left[\frac{1}{3}x^3\right]_{-2r}^{0} + \pi r\left[\frac{x^2}{2}\right]_{-2r}^{0} + \pi r^2\left[x\right]_{-2r}^{0} + \pi r^2\left[x\right]_{0}^{r} - \pi\left[\frac{1}{3}x^3\right]_{0}^{r}$

$V = \frac{\pi}{12}(0 + 8r^3) + \frac{\pi r}{2}(0 - 4r^2) + \pi r^2(0 + 2r) + \pi r^2(r - 0) - \frac{\pi}{3}(r^3 - 0)$

$V = \frac{4}{3}\pi r^3$ u. v.