# Corrigé (GD juin 2012)

**I** 1) $3+2\cdot 5^{1-x} \geq 5^x \iff 3+2\cdot 5\cdot 5^{-x} \geq 5^x \quad |\cdot 5^x>0$

$\iff 3\cdot 5^x+10 \geq 5^{2x} \quad (*)$

posons $y=5^x$, alors $(*)$ devient :

$3y+10 \geq y^2 \iff y^2-3y-10 \leq 0$

| $y$ | | $-2$ | | $5$ | |
|---|---|---|---|---|---|
| $y^2-3y-10$ | $+$ | $0$ | $-$ | $0$ | $+$ |

$\Delta = 9+40=49$

$y' = \frac{3+7}{2}=5$

$y'' = \frac{3-7}{2} = -2$

$\iff -2 \leq y \leq 5$

$\iff \underbrace{-2 \leq 5^x}_{\text{vrai } \forall x\in\mathbb{R}} \leq 5$

$\iff 5^x \leq 5^1$

$\iff x \leq 1$

$S = [1 ; -\infty[$

2) $\log_{\sqrt{2}}(1-2x) + \log_{\frac{1}{2}}(x+7) = 0$

C.E. $\begin{cases} 1-2x>0 \\ x+7>0 \end{cases} \iff \begin{cases} x<\frac{1}{2} \\ x>-7 \end{cases}$ $D_E = ]-7, \frac{1}{2}[$

$\iff \frac{\ln(1-2x)}{\ln\sqrt{2}} + \frac{\ln(x+7)}{\ln\frac{1}{2}} = 0$

$\iff \frac{\ln(1-2x)}{\frac{1}{2}\ln 2} + \frac{\ln(x+7)}{-\ln 2} = 0 \quad |\cdot \ln 2>0$

$\iff 2\ln(1-2x) - \ln(x+7) = 0$

$\iff \cancel{\ln}(1-2x)^2 = \cancel{\ln}(x+7)$

$\iff 1-4x+4x^2-x-7=0$

$\iff 4x^2-5x-6=0$

$\Delta = 25+96=121$, $x' = \frac{5+11}{8} = 2 \notin D_E$, $x'' = \frac{5-11}{8} = -\frac{3}{4} \in D_E$

$S = \{-\frac{3}{4}\}$

**II** 1) $\ell = \lim\limits_{+\infty} \left(\frac{-1-x}{1-x}\right)^{1-x}$ $\quad 1-x \to -\infty$ $\quad$ f.i. $1^\infty$

$\approx \frac{-x}{-x} \to 1$

posons $\frac{-1-x}{1-x} = 1+y$, alors :

- $y = \frac{-1-x}{1-x} - 1 = \frac{-1-x-1+x}{1-x} = \frac{-2}{1-x} = \frac{2}{x-1}$
- $x \to +\infty$ ssi $y \to 0$
- $y = \frac{2}{x-1} \iff x-1 = \frac{2}{y} \quad |\cdot(-1) \iff 1-x = -\frac{2}{y}$

D'où $l = \lim\limits_{y \to 0} (1+y)^{-\frac{2}{y}} = \lim\limits_{0} \Big[\underbrace{(1+y)^{\frac{1}{y}}}_{\to e}\Big]^{-2} = e^{-2}$

2) $\lim\limits_{-\infty} 10^{x-1} \cdot \log(1-x) = \lim\limits_{-\infty} \underbrace{e^{\overbrace{(x-1)\ln 10}^{\to -\infty}}}_{\to 0} \cdot \dfrac{\overbrace{\ln(1-x)}^{\to +\infty}}{\ln 10}$ f.i. $0 \cdot \infty$

$= \lim\limits_{-\infty} \dfrac{1}{\ln 10} \cdot \dfrac{\overbrace{\ln(1-x)}^{\to +\infty}}{\underbrace{e^{(1-x)\ln 10}}_{\to +\infty}}$ f.i. $\dfrac{\infty}{\infty}$

$\underset{(H)}{=} \lim\limits_{-\infty} \dfrac{1}{\ln 10} \cdot \dfrac{\frac{-1}{1-x}}{-\ln 10 \cdot e^{(1-x)\ln 10}}$

$= \lim\limits_{-\infty} \dfrac{1}{\ln^2 10 \cdot \underbrace{(1-x)}_{\to +\infty} \cdot \underbrace{e^{(1-x)\ln 10}}_{\to +\infty}}$

$= \left(\dfrac{1}{+\infty}\right)$

$= 0$

III $f(x) = (2x-1)e^{x-1} + 4$

1) $D_f = \mathbb{R}$

- $\lim\limits_{+\infty} \underbrace{(2x-1)}_{\to +\infty} \overbrace{e^{x-1}}^{\to +\infty} + 4 = +\infty$, pas d'A.H.D.
- $\lim\limits_{+\infty} \dfrac{f(x)}{x} = \lim\limits_{+\infty} \Big( \underbrace{\dfrac{2x-1}{x}}_{\approx \frac{2x}{x} \to 2} \cdot \underbrace{e^{x-1}}_{+\infty} + \underbrace{\dfrac{4}{x}}_{0} \Big) = +\infty$, branche parab. de direction $(Oy)$
- $\lim\limits_{-\infty} \underbrace{(2x-1)}_{\to -\infty} \underbrace{e^{x-1}}_{0} + 4$ f.i. $\infty \cdot 0$

$= \lim\limits_{-\infty} \Big( \dfrac{\overbrace{2x-1}^{\to -\infty}}{\underbrace{e^{1-x}}_{\to +\infty}} + 4 \Big)$ f.i. $\dfrac{\infty}{\infty}$

$\underset{(H)}{=} \lim\limits_{-\infty} \Big( \dfrac{2}{-e^{1-x}} + 4 \Big)$

$= \left(\dfrac{2}{-\infty}\right) + 4$

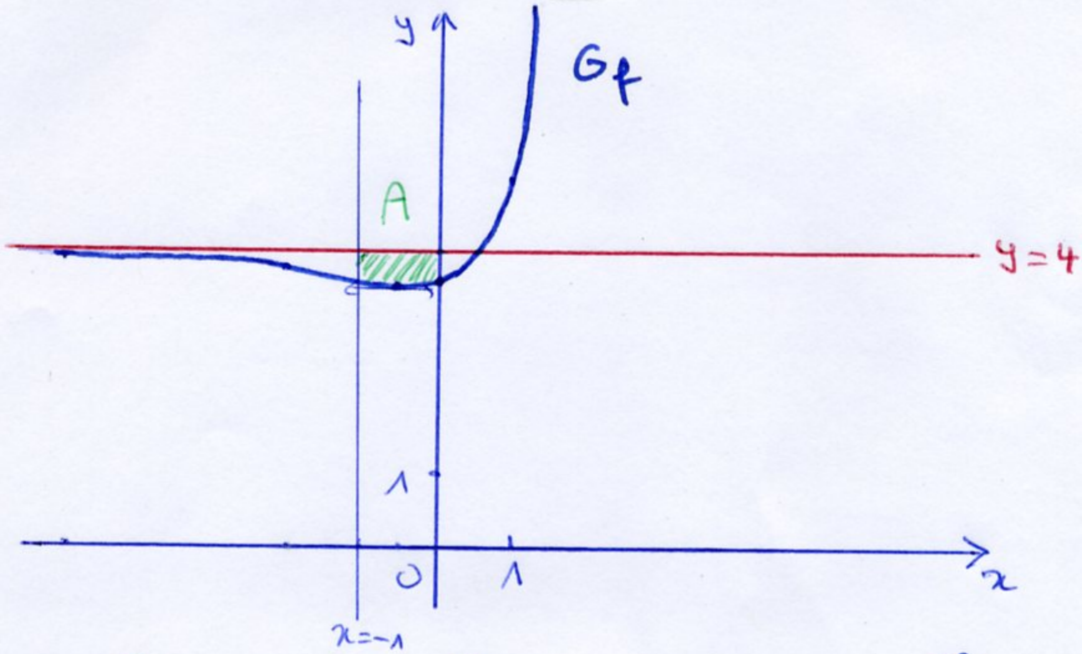
$= 4$ A.H.G. $y = 4$

2) $f'(x) = 2\cdot e^{x-1} + (2x-1)\cdot e^{x-1} = e^{x-1}(2+2x-1)$

$= (2x+1)e^{x+1}$ signe de $2x+1$

| $x$ | $-\infty$ | | $-\frac{1}{2}$ | | $+\infty$ |
|---|---|---|---|---|---|
| $f'(x)$ | | $-$ | $0$ | $+$ | |
| $f$ | $4$ | $\searrow$ | $-2e^{-3/2}+4 \approx 3,6$ | $\nearrow$ | $+\infty$ |

3)




4) $A = \int_{-1}^{0} (4 - f(x))\,dx = \int_{-1}^{0} \cancel{4} - (2x-1)e^{x-1} \cancel{-4}\,dx = \int_{-1}^{0} \underbrace{(1-2x)e^{x-1}}_{g(x)}\,dx$

i.p.p. : $u = 1-2x \quad u' = -2$

$v' = e^{x-1} \quad v = e^{x-1}$

$G(x) = (1-2x)e^{x-1} + 2\int e^{x-1}dx = (1-2x)e^{x-1} + 2\cdot e^{x-1}$

$= (3-2x)e^{x-1}$

$A = G(0) - G(-1) = 3\cdot e^{-1} - 5\cdot e^{-2}$ u.a. $\approx 0,43$ u.a.

IV $f(x) = x - \ln(1-x)$ C.E. $1-x>0 \Leftrightarrow x<1$, $D_f = ]1,-\infty[ = D_{f'} = D_{f''}$

1) $(T) \equiv y - f(-1) = f'(-1)\cdot(x+1)$

$f(-1) = -1 - \ln 2$

$f'(x) = 1 - \frac{-1}{1-x} = 1 + \frac{1}{1-x} = \frac{1-x+1}{1-x} = \frac{2-x}{1-x}$

$f'(-1) = \frac{3}{2}$

D'où : $(T) \equiv y = \frac{3}{2}(x+1) - 1 - \ln 2 \equiv \underline{y = \frac{3}{2}x + \frac{1}{2} - \ln 2}$

2) $f''(x) = \dfrac{-1(1-x)-(-x)(2-x)}{(1-x)^2} = \dfrac{-1+x+2-x}{(1-x)^2} = \dfrac{1}{(1-x)^2} > 0$

donc $G_f$ est convexe sur $D_f = ]1,-\infty[$

V 1) a) $I = \int_1^e \underbrace{(2x^2-1)\ln x}_{f(x)}\, dx$

i.p.p. $u = \ln x \quad u' = \dfrac{1}{x}$

$v' = 2x^2-1 \quad v = \dfrac{2}{3}x^3 - x$

$F(x) = \left(\dfrac{2}{3}x^3 - x\right)\ln x - \int\left(\dfrac{2}{3}x^2 - 1\right) dx$

$= \left(\dfrac{2}{3}x^3 - x\right)\ln x - \dfrac{2}{9}x^3 + x$

$F(e) = \dfrac{2}{3}e^3 \cancel{-e} - \dfrac{2}{9}e^3 \cancel{+e} = \dfrac{4}{9}e^3$

$F(1) = -\dfrac{2}{9} + 1 = \dfrac{7}{9}$

$I = F(e) - F(1) = \dfrac{4}{9}e^3 - \dfrac{7}{9}$

b) $\displaystyle\int \frac{1-x}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{1-(2x)^2}}\,dx + \int \frac{-x}{\sqrt{1-4x^2}}\,dx$

$\begin{cases} u = 2x \\ u' = 2 \Leftrightarrow \frac{1}{2}u' = 1 \\ f = \frac{1}{2}\frac{u'}{\sqrt{1-u^2}} \\ F = \frac{1}{2}\operatorname{Asin} u \end{cases}$ $\qquad$ $\begin{matrix} v = 1-4x^2 \\ v' = -8x \Leftrightarrow \frac{1}{8}v' = -x \\ g = \frac{1}{8}\frac{v'}{v^{\frac{1}{2}}} = \frac{1}{8}v^{-\frac{1}{2}}\cdot v' \\ G = \frac{1}{8}\cdot\frac{1}{\frac{1}{2}}v^{\frac{1}{2}} = \frac{\sqrt{v}}{4} \end{matrix}$

$= \dfrac{1}{2}\operatorname{Asin} 2x + \dfrac{\sqrt{1-4x^2}}{4} + c$

2) $f(x) = \dfrac{(1-\ln x)^3}{x}$

posons : $u = 1 - \ln x$

alors : $u' = -\dfrac{1}{x}$

$f = -u' \cdot u^3$

$F = -\dfrac{1}{4}u^4 + k$

$F(x) = -\dfrac{1}{4}(1-\ln x)^4 + k$

$F(e^2) = 0 \Leftrightarrow -\dfrac{1}{4}(1-2)^4 + k = 0 \Leftrightarrow k = \dfrac{1}{4}\cdot 1 = \dfrac{1}{4}$

$F(x) = -\dfrac{1}{4}(1-\ln x)^4 + \dfrac{1}{4}$

# Problème

1) $A = \int_0^6 \big(n(x) - s(x)\big)\,dx \overset{V200}{\cong} 20{,}31$ u.a.

$1$ u.a. $= 100^2 = 10'000\ m^2 = 1$ ha

d'où $A \simeq 20{,}31$ ha $(= 20{,}31 \cdot 10^4\ m^2)$

Soit $h$ la profondeur <u>moyenne</u> du lac, alors:

$$V = A \cdot h \Leftrightarrow h = \frac{V}{A}$$

$$h \simeq \frac{3{,}5 \cdot 10^{\cancel{8}}}{20{,}31 \cdot \cancel{10^4}} \simeq 1{,}72\ m$$

2) a) $s(x)$ minimale pour $x = \frac{11}{2} = 5{,}5$ 
$s(5{,}5) \simeq 0{,}79$ $\Big\}$ donc $S(5{,}5\,;\,0{,}79)$

$m(x)$ maximale pour $x \simeq 5{,}26$ 
$m(5{,}26) \simeq 9{,}52$ $\Big\}$ donc $N(5{,}26\,;\,9{,}52)$

b) $d \simeq \sqrt{(5{,}5 - 5{,}26)^2 + (0{,}79 - 9{,}52)^2} \simeq 8{,}7333$ u.l.
$\simeq$ <u>873,33 m</u>

c) $d \simeq 0{,}87333$ km donc $T = \frac{d}{v} \simeq 0{,}09704$ h
$\simeq 349''$
$\simeq 5'49''$

d) distance de S jusqu'au mur $= \int_{5{,}5}^{6} \sqrt{1 + (s'(x))^2}\,dx$

$\simeq 1{,}3725$ u.l $\simeq 137{,}25$ m

longueur du parcours le long du mur $= m(6) - s(6)$
$\simeq 5{,}4588$ u.l.
$\simeq 545{,}88$ m

distance du mur jusqu'à N $= \int_{5{,}26}^{6} \sqrt{1 + (n'(x))^2}\,dx$

$\simeq 2{,}2776$ u.l $\simeq 227{,}76$ m

parcours total du coureur $\simeq 137{,}25 + 545{,}88 + 227{,}76$
$\simeq 910{,}89$ m $(\simeq 0{,}91089$ km$)$

durée de la course $\simeq \frac{0{,}91089}{10} \cdot 3600 \simeq 328'' < T$ !

Conclusion: Le coureur arrive 349-328 = 21' avant le bâteau!

3) Appelons $c$ l'abscisse du point C : $C(c, m(c))$

comme t est la tangente à $\mathcal{C}_m$ au point C on a :

$$t \equiv y = m'(c)\,(x-c) + m(c)$$

Or $K(1,4) \in t$ donc $\underbrace{4 = m'(c)(1-c) + m(c)}_{\text{équation d'inconnue } c \text{ à résoudre}}$

V200 : $c \simeq 4{,}93$

D'où $C(4{,}93 ; 9{,}27)$

$$t \equiv y = 1{,}34\,(x - 4{,}93) + 9{,}27$$

$$\equiv y = 1{,}34\,x + 2{,}66$$