Corrigé modèle

I. 1) a) $\log_{x+4} 16 + 4 = \log_2 \frac{0{,}25}{(x+4)^2}$ $\quad D =]-4;-3[\cup]-3;+\infty[$

Posons : $\ln(x+4) = t$

$\frac{4\ln 2}{\ln(x+4)} + 4 = \frac{-2\ln 2 - 2\ln(x+4)}{\ln 2} \quad \Big| \cdot \ln 2 \cdot \ln(x+4)$

$4(\ln 2)^2 + 4\ln 2 \, t = t(-2\ln 2 - 2t) \Leftrightarrow t^2 + 3\cdot\ln 2 \, t + 2(\ln 2)^2 = 0 \quad \Delta = (\ln 2)^2$

$t = \frac{-3\ln 2 - \ln 2}{2} = \ln\frac{1}{4} = \ln(x+4) \Leftrightarrow x = -\frac{15}{4}$ ($\in D$) ou $t = \frac{-3\ln 2 + \ln 2}{2} = \ln\frac{1}{2} = \ln(x+4) \Leftrightarrow x = -\frac{7}{2}$ ($\in D$)

$\Rightarrow S = \{-\frac{15}{4}; -\frac{7}{2}\}$

b) $4^x - 3^{x+\frac{1}{2}} = 3^{x-\frac{1}{2}} - \underbrace{2^{2x}}_{=4^x} \Leftrightarrow 2\cdot 4^x = \underbrace{\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)}_{=\frac{4\sqrt{3}}{3}} \cdot 3^x \Leftrightarrow \left(\frac{4}{3}\right)^x = \frac{2\sqrt{3}}{3} = \left(\frac{4}{3}\right)^{\frac{1}{2}} \Leftrightarrow x = \frac{1}{2}$

$S = \{\frac{1}{2}\}$

2) (E) $e^x + 2(a-1)e^{-x} = 2a-1 \quad | \cdot e^x$

$e^{2x} - (2a-1)e^x + 2(a-1) = 0 \quad \Delta = (2a-1)^2 - 8(a-1) = (2a-3)^2 \geq 0$

<u>1er cas :</u> $\Delta = 0 \Leftrightarrow a = \frac{3}{2}$

$e^x = \frac{3-1}{2} = 1 \Leftrightarrow x = 0$

$S = \{0\}$

<u>2e cas :</u> $\Delta > 0 \Leftrightarrow a \neq \frac{3}{2}$

$e^x = \frac{2a-1+2a-3}{2} = 2(a-1)$ ou $e^x = \frac{2a-1-2a+3}{2} = 1$

Si $a > 1$ ($a \neq \frac{3}{2}$), (E) $\Leftrightarrow x = 0$ ou $x = \ln(2(a-1))$ $\quad S = \{0; \ln(2(a-1))\}$

Si $a \leq 1$, (E) $\Leftrightarrow x = 0$ et $S = \{0\}$

II 1) $\lim\limits_{x\to+\infty} \left(\frac{4-2x}{1-2x}\right)^{3-2x} = \lim\limits_{x\to+\infty} \left(1 + \frac{-3}{2x-1}\right)^{3-2x}$

Posons : $\frac{-3}{2x-1} = t \Leftrightarrow x = \frac{1}{2} - \frac{3}{2t}$

Si $x \to +\infty$, alors $t \to 0$.

$= \lim\limits_{t\to 0} (1+t)^{3-1+\frac{3}{t}} = \lim\limits_{t\to 0} \Big[\underbrace{(1+t)^{\frac{1}{t}}}_{\to e}\Big]^3 \underbrace{(1+t)^2}_{\to 1} = e^3$

2) $\lim\limits_{0} \left(\frac{x^2+2}{x^2}\right)^{\sin x} = \lim\limits_{0} e^{\overbrace{\sin x \cdot \ln\frac{x^2+2}{x^2}}^{\to 0}} = 1$

$\lim\limits_{0} \frac{\ln\frac{x^2+2}{x^2}}{\frac{1}{\sin x}} = \lim\limits_{0} \frac{\frac{x^2}{x^2+2} \cdot \frac{2x^3 - (x^2+2)2x}{x^4}}{-\frac{\cos x}{\sin^2 x}}$

$= \lim\limits_{0} \frac{4x}{(x^2+2)x^3} \cdot \frac{\sin^2 x}{\cos x} = \lim\limits_{0} \underbrace{\frac{4x}{x^2+2}}_{\to 0} \cdot \underbrace{\left(\frac{\sin x}{x}\right)^2}_{\to 1} \cdot \underbrace{\frac{1}{\cos x}}_{\to 1} = 0$

III 1) $g(x) = e^x - 1 - x \quad \forall x \in \mathbb{R} : g'(x) = e^x - 1 \; : \; g'(x) \geq 0 \Leftrightarrow x \geq 0$

x		0	
$g'(x)$	$-$	0	$+$
$g(x)$	↘	0	↗

$g(0) = 0$ Donc :

x		0	
$g(x)$	$+$	0	$+$

2) $f(x) = \begin{cases} \frac{xe^x}{e^x-1} & \text{si } x \neq 0 \\ 1 & \text{si } x = 0 \end{cases} \quad \text{dom} f = \mathbb{R}$

a) $\lim\limits_{x\to 0} f(x) = \lim\limits_{0} \frac{xe^x}{e^x-1} \underset{(H)}{=} \lim\limits_{0} \frac{e^x + xe^x}{e^x} = \lim\limits_{0} (1+x) = 1 = f(0) \Rightarrow f$ continue en 0

$\lim\limits_{0} \frac{f(x)-f(0)}{x} = \lim\limits_{0} \frac{\frac{xe^x}{e^x-1} - 1}{x} = \lim\limits_{0} \frac{xe^x - e^x + 1}{x(e^x-1)} \underset{(H)}{=} \lim\limits_{0} \frac{xe^x + \cancel{e^x} - \cancel{e^x}}{e^x - 1 + xe^x} \underset{(H)}{=} \lim\limits_{0} \frac{xe^x + e^x}{e^x + e^x + xe^x}$

$= \lim\limits_{0} \frac{x+1}{x+2} = \frac{1}{2} \Rightarrow f$ dérivable en 0 et $f'(0) = \frac{1}{2}$.

b) $\lim\limits_{-\infty} f(x) = \lim\limits_{-\infty} \frac{\overbrace{(xe^x)}^{\to 0}}{\underbrace{e^x - 1}_{\to -1}} \quad \lim\limits_{-\infty} xe^x = \lim\limits_{-\infty} \frac{x}{e^{-x}} \underset{(H)}{=} \lim\limits_{-\infty} \frac{1}{-e^{-x}} = 0$

$= 0 \Rightarrow y = 0$ AH en $-\infty$

$\lim\limits_{+\infty} f(x) = \lim\limits_{+\infty} \dfrac{xe^{x}}{\underbrace{e^x-1}_{+\infty}} = \lim\limits_{+\infty} \dfrac{xe^x}{e^x\underbrace{(1-e^{-x})}_{\to 1}} = +\infty$

$\lim\limits_{+\infty} \dfrac{f(x)}{x} = \lim\limits_{+\infty} \dfrac{e^x}{e^x-1} = \lim\limits_{+\infty} \dfrac{e^x}{e^x(1-e^{-x})} = 1$

$\lim\limits_{+\infty} [f(x)-x] = \lim\limits_{+\infty} \dfrac{xe^x - xe^x + x}{e^x-1} \underset{\text{H}}{=} \lim\limits_{+\infty} \dfrac{1}{e^x} = 0$

$\Rightarrow y = x$ AO en $+\infty$.

Rq. : $f(x) = \dfrac{x(e^x-1)+x}{e^x-1} = x + \dfrac{x}{e^x-1}$

et $\lim\limits_{+\infty} \dfrac{x}{e^x-1} = 0 \Rightarrow y = x$ AO en $+\infty$.

Position : Posons : $\forall x \neq 0$ $r(x) = y_{C_f} - y_{AO} = \dfrac{x}{e^x-1}$

Remarquons que : $f(0) = 1$

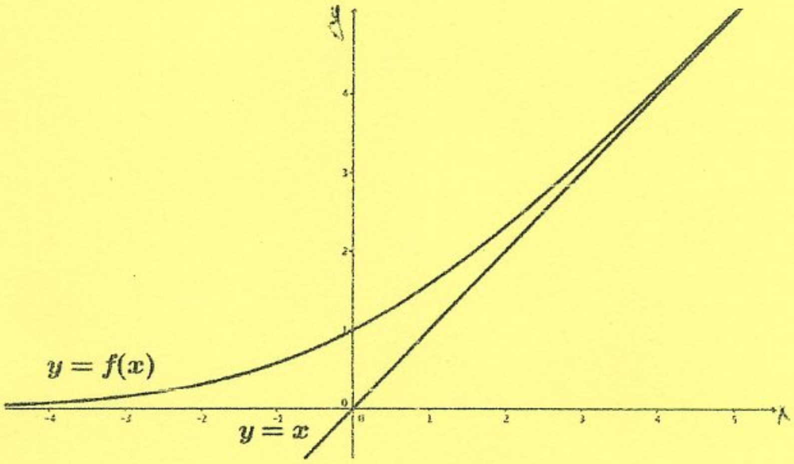
$\Rightarrow C_f/AO$ en 0 (car $(0,0) \in AO$)

x		0	
x	$-$	0	$+$
e^x-1	$-$	0	$+$
$r(x)$	$+$	$\Vert$	$+$
	C_f/AO		C_f/AO

c) $\forall x \neq 0$: $f'(x) = \dfrac{(e^x-1)(e^x+xe^x) - xe^x e^x}{(e^x-1)^2} = \dfrac{e^x(e^x-1-x)}{(e^x-1)^2} = \dfrac{e^x g(x)}{(e^x-1)^2} > 0$

x	$-\infty$		$+\infty$
$f'(x)$		$+$	
$f(x)$	0	$\nearrow$	$+\infty$

d) Graphique :



IV) 1) a) $\dfrac{3}{x^3-1} = \dfrac{3}{(x-1)(x^2+x+1)} = \dfrac{a}{x-1} + \dfrac{bx+c}{x^2+x+1} = \dfrac{(a+b)x^2+(a-b+c)x+a-c}{x^3-1}$ $\begin{cases} a+b=0 \\ a-b+c=0 \\ a-c=3 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=-1 \\ c=-2 \end{cases}$

$I = \displaystyle\int \frac{3}{x^3-1}\,dx = \int\left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1}\right)dx = \ln|x-1| - \frac{1}{2}\underbrace{\int \frac{2x+1+3}{x^2+x+1}\,dx}_{=J}$

$J = \displaystyle\int \frac{2x+1}{x^2+x+1}\,dx + \int \frac{3}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \ln(x^2+x+1) + \frac{4}{3}\,3\,\frac{\sqrt{3}}{2}\int \frac{\frac{2}{\sqrt{3}}}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}\,dx = \ln(x^2+x+1) + 2\sqrt{3}\,\mathrm{Arctan}\,\frac{2x+1}{\sqrt{3}}$

Finalement : $I = \ln|x-1| - \frac{1}{2}\ln(x^2+x+1) - \sqrt{3}\,\mathrm{Arctan}\,\dfrac{2x+1}{\sqrt{3}} + k$, sur $I \subset \mathbb{R}\setminus\{1\}$ (intervalle !)

b) $\displaystyle\int \frac{2}{e^x - e^{-x}}\,dx$ Posons $e^x = t \Leftrightarrow dx = \frac{1}{t}\,dt$

$= \displaystyle\int \frac{2}{t-\frac{1}{t}}\cdot\frac{1}{t}\,dt = \int \frac{2}{t^2-1}\,dt$ $\quad \dfrac{2}{t^2-1} = \dfrac{a}{t-1} + \dfrac{b}{t+1} = \dfrac{(a+b)t+a-b}{t^2-1}$ $\begin{cases} a+b=0 \\ a-b=2 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=-1 \end{cases}$

$= \displaystyle\int\left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt = \ln|t-1| - \ln(t+1) + k = \ln|e^x-1| - \ln(e^x+1) + k$, sur $I \subset \mathbb{R}^*$

c) $I = \displaystyle\int \cos(\ln x)\,dx$ $\quad u = \cos(\ln x) \Rightarrow u' = -\frac{1}{x}\sin(\ln x)$, $v' = 1 \Rightarrow v = x$

$= x\cos(\ln x) + \displaystyle\int \sin(\ln x)\,dx$ $\quad u = \sin(\ln x) \Rightarrow u' = \frac{1}{x}\cos(\ln x)$, $v' = 1 \Rightarrow v = x$

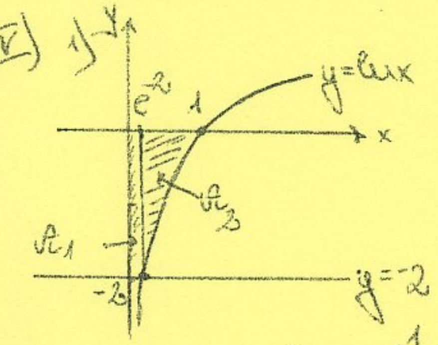
$= x(\cos(\ln x) + \sin(\ln x)) - I + k \Rightarrow I = \frac{1}{2}x(\cos(\ln x) + \sin(\ln x)) + k$, sur $I \subset \mathbb{R}_+^*$

2) a) Posons : $z = \cos x + i\sin x$

$$\cos^6 x = \left(\frac{z+\bar z}{2}\right)^6 = \frac{1}{64}\left(z^6 + 6z^4 + 15z^2 + 20 + 15\bar z^2 + 6\bar z^4 + \bar z^6\right)$$
$$= \frac{1}{32}\left(\cos 6x + 6\cos 4x + 15\cos 2x + 10\right)$$
$$\Rightarrow \cos^6 x = \frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x + \frac{15}{32}\cos 2x + \frac{5}{16}$$

b) $$\int (1+\sin x)^4(1-\sin x)^3\,dx = \int (1-\sin^2 x)^3(1+\sin x)\,dx = \int \cos^6 x\,(1+\sin x)\,dx$$
$$= \int\left(\cos^6 x + \cos^6 x\cdot\sin x\right)dx = \frac{1}{192}\sin 6x + \frac{3}{64}\sin 4x + \frac{15}{64}\sin 2x + \frac{5}{16}x - \frac{1}{7}\cos^7 x + k,$$ sur $I \subset \mathbb{R}$

V) 1)



$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 = 2\cdot e^{-2} - \int_{e^{-2}}^{1} \ln x\,dx$$ $u = \ln x \Rightarrow u' = \frac{1}{x}$, $v' = 1 \Rightarrow v = x$

$$= 2e^{-2} - \left[x\ln x - x\right]_{e^{-2}}^{1}$$
$$= 2e^{-2} - \left(-1 - (-2e^{-2} - e^{-2})\right) = (1 - e^{-2})\ \text{u.a.}$$

2) $$V = \underbrace{\pi\cdot 4\cdot e^{-2}}_{\text{cylindre}} + \pi\int_{e^{-2}}^{1} \ln^2 x\,dx$$ $u = \ln^2 x \Rightarrow u' = \frac{2\ln x}{x}$, $v' = 1 \Rightarrow v = x$

$$= 4\pi e^{-2} + \pi\left(\left[x\ln^2 x\right]_{e^{-2}}^{1} - 2\int_{e^{-2}}^{1}\ln x\,dx\right) = 4\pi e^{-2} + \pi\left(-4e^{-2} - 2\left[x\ln x - x\right]_{e^{-2}}^{1}\right)$$
$$= 4\pi e^{-2} - 4\pi e^{-2} - 2\pi\left(-1 - (-2e^{-2} - e^{-2})\right) = 2\pi\left(1 - 3e^{-2}\right)\ \text{u.v.}$$